<u>How to solve differential equations complete</u>.

Author:
mathematics teacher Zbigniew Jan Stebel

Find a solution to the equation:

$$(1)\ 3y+e^t+(3t+\cos y)\frac{dy}{dt}=0$$

<u>Method One:</u>

In this case

$M(t,y)=3y+e^t,$

$N(t,y)=3t+\cos y.$

The equation is the total equation, because:

$$\frac{\partial M}{\partial y}=3$$

$$\frac{\partial N}{\partial t}=3,$$

so $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial t}.$

Therefore, there is a function $\Phi(t,y)$ such that the conditions

i. $\dfrac{\partial \Phi}{\partial t}=M(t,y)=3y+e^t$

ii. $\dfrac{\partial \phi}{\partial y}=N(t,y)=3t+\cos y$

Go to function $\Phi(t,y)$:

$$\Phi(t,y)=\int(3y+e^t)dt+h(y)=3yt+e^t+h(y).$$

Differentiating terms of variable y and get:

$$\frac{\partial \Phi}{\partial y}=3t+h'(y).$$

Comparing the parties last equation to the equation (ii) we have:

$3t+\cos(y)=3t+h'(y),$ so $h'(y)=\cos(y)\Rightarrow h(y)=\sin(y),$

hence we get

$\Phi(t,y)=3yt+e^t+\sin(y).$

The equation (1) has a solution

$3yt+e^t+\sin(y)=C.$

<u>Method Two:</u>

If $N(t,y)=\dfrac{\partial \Phi}{\partial y}$ that course $\Phi(t,y)=\int N(t,y)dy+k(t).$

Because $M(t,y)=\dfrac{\partial \Phi}{\partial t}=\displaystyle\int\frac{\partial N(t,y)}{\partial t}dy+k'(t)$, Thus $k'(t)$ it can be found from the equation:

$$k'(t)=M(t,y)-\int\frac{\partial N(t,y)}{\partial t}dy.$$

Inserting the functions get:

$\Phi(t,y)=\int(3t+\cos y)dy+k(t)=3ty+\sin y+k(t).$

Differentiating the variable t and taking into account the condition (i):

$3y+k'(t)=3y+e^t$, hence $k(t)=e^t$.

Finally, we get:

$\Phi(t,y)=3ty+\sin y+e^t$, that is, the same pattern as in the first method.

We have to resolve the issue of the original:

$$(2)\quad \begin{cases} 4t^3e^{t+y}+t^4e^{t+y}+2t+(t^4e^{t+y}+2y)\dfrac{dy}{dt}=0 \\ y(0)=1 \end{cases}$$

At the top mark the appropriate functions

$M(t,y)=4t^3e^{t+y}+t^4e^{t+y}+2t,$

$N(t,y)=t^4e^{t+y}+2y.$

Is this a complete equation?

$$\frac{\partial M}{\partial y}=(4t^3+t^4)e^{t+y}$$

$$\frac{\partial N}{\partial t}=(4t^3+t^4)e^{t+y},$$

that is, the equation is complete, because $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial t}.$

There is a function of $\Phi(t,y)$ satisfying the conditions:

$$\frac{\partial \Phi}{\partial t}=4t^3e^{t+y}+t^4e^{t+y}+2t$$

$$\frac{\partial \Phi}{\partial y}=t^4e^{t+y}+2y$$

It's easier to calculate the equation over another, thus taking advantage of this equation we get:

$\Phi(t,y)=t^4e^{t+y}+y^2+k(t).$

Differentiating the last variable to the equation after t get:

$$\frac{\partial \Phi}{\partial t}=(t^4+4t^3)e^{t+y}+k'(t).$$

Comparing with the condition receive the first equation of the form:

$k'(t)=2t \Rightarrow k(t)=t^2.$

Thus, a total solution:

$\Phi(t,y)=t^4e^{t+y}+y^2+t^2=C,$ where $C-$ integration has become.

With the initial conditions, we know that

$t=0, y=1 \Rightarrow C=1.$

We received a solution with the initial conditions:

$t^4e^{t+y}+y^2+t^2=1.$

We have a differential equation:

$$M(t,y)+N(t,y)\frac{dy}{dt}=0$$

Which is not an absolute equation?
Can I turn them into a complete equation?

$$\mu(t,y)M(t,y)+\mu(t,y)N(t,y)\frac{dy}{dt}=0(*)$$

When this equation is the total equation?

$$\frac{\partial}{\partial y}\left(\mu(t,y)M(t,y)\right)=\frac{\partial}{\partial t}\left(\mu(t,y)N(t,y)\right)$$

$$\Downarrow$$

$$M\frac{\partial \mu}{\partial y}+\mu\frac{\partial M}{\partial y}=N\frac{\partial \mu}{\partial t}+\mu\frac{\partial N}{\partial t}(**)$$

The equation (*) is the total equation and then only if the equation is satisfied (**).

<u>Definition</u>
Feature μ satisfies the equation (**) is called a factor which is calculated over the equation (*).
Unfortunately, only in specific situations, we can solve the equation (**).
We know how to resolve them when the function μ is a function only argument t, or only argument y.
When $\mu(t,y)=\mu(t)$ then the equation (**) reduces to the equation:

$$\mu\frac{\partial M}{\partial y}=N\frac{\partial \mu}{\partial t}+\mu\frac{\partial N}{\partial t}\Rightarrow N\frac{\partial \mu}{\partial t}=\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right).$$

Therefore: $\dfrac{\partial \mu}{\partial t}=\dfrac{\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)}{N}$.

This equation makes sense when: $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}}{N}$) is a function only argument t, and therefore we can write:

$$R(t)=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}}{N}.$$

In this case we receive the order:

$$\frac{\partial \mu}{\partial t}=\mu R(t)\Rightarrow\int\frac{d\mu}{\mu}=\int R(t)dt\Rightarrow\ln(\mu(t))=\int R(t)dt\Rightarrow\mu(t)=\exp\left(\int T=R(t)dt\right).$$

We received that: $\mu(t)=e^{\int R(t)dt}$.
The data is a differential equation:

$$(3)\ \frac{y^2}{2}+2ye^t+(y+e^t)\frac{dy}{dt}=0.$$

Consider whether it is a complete differential equation?

Let $M(t,y)=\dfrac{y^2}{2}+2ye^t$,
$N(t,y)=y+e^t$.

Then $\begin{cases} \dfrac{\partial M}{\partial y} = y + 2e^{t} \\ \dfrac{\partial N}{\partial t} = e^{t}, \end{cases}$ so $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial t}$ therefore differential equation is not complete.

According to the algorithm, we have

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial t}}{N} = \frac{y + e^{t}}{y + e^{t}} = 1 = R(t),$$ thence $\mu(t) = e^{\int 1dt} = e^{t}$, is this factor.

So the equation of the form:

$$\frac{y^2}{2}e^{t} + 2ye^{t} + \left(ye^{t} + e^{2t}\right)\frac{dy}{dt} = 0$$

is an absolute equation.

We look for such functions $\Phi(t, y)$ that the conditions are met:

$$\frac{\partial \Phi}{\partial t} = \frac{y^2}{2}e^{t} + 2ye^{2t}$$

$$\frac{\partial \phi}{\partial y} = ye^{t} + e^{2t}.$$

We count on both sides of the integrated two equations, respectively terms of variable t and y:

$$\Phi(t, y) = \frac{y^2}{2}e^{t} + ye^{2t} + h(y),$$

$$\Phi(t, y) = \frac{y^2}{2}e^{t} + ye^{2t} + k(t).$$

Comparing these parties to the equation, we see that $h(y) = 0 \Leftrightarrow k(t) = 0.$

Therefore, the solution of our equation is the form:

$$\Phi(t, y) = \frac{y^2}{2}e^{t} + ye^{2t} = C.$$

Assume further that the

(4) $y(0) = 1.$

Then we receive:

$$\Phi(0,1) = \frac{1}{2}e^{0} + e^{0} = \frac{1}{2} + 1 = \frac{3}{2} = C.$$

Ultimately, the solution differential equation (3) from the initial condition (4) is:

$$\frac{y^2}{2}e^{t} + ye^{2t} = \frac{3}{2}.$$

Complete differential equations can also solve other alternative methods, for example, the method operators.